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LETTER TO THE EDITOR

The quantum symmetric *XXZ* chain at $\Delta = -\frac{1}{2}$, alternating-sign matrices and plane partitions

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Abstract

We consider the ground-state wavefunction of the quantum symmetric antiferromagnetic *XXZ* chain with open and twisted boundary conditions at $\Delta = -\frac{1}{2}$, along with the ground-state wavefunction of the corresponding O(n) loop model at $n = 1$. Based on exact results for finite-size systems, sums involving the wavefunction components, and in some cases the largest component itself, are conjectured to be directly related to the total number of alternating-sign matrices and plane partitions in certain symmetry classes.

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Very recently Razumov and Stroganov [1] have made some remarkable conjectures in which the number of $n \times n$ alternating-sign matrices appear in certain properties of the ground-state wavefunction of the antiferromagnetic *XXZ* Heisenberg chain at $\Delta = -\frac{1}{2}$ defined on an odd number of sites L. At this point, the ground-state energy of the *XXZ* Hamiltonian

$$H = -\tfrac{1}{2}\sum_{j=1}^{L}\left(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta\sigma_j^z\sigma_{j+1}^z\right) \tag{1}$$

with periodic boundary conditions, is simply [2, 3]

$$E_0 = -3L/4. \tag{2}$$

The ground state is twofold degenerate with spin $S^z = \pm\frac{1}{2}$. Define the p-sum $S_L^{(p)}$ of the components of the ground-state wavefunction ψ_0 by

$$S_L^{(p)} = \sum_{s=1}^{\binom{L}{[L/2]}} \psi_0(s)^p \tag{3}$$

where $[L/2]$ defines the integer part of $L/2$ and $\binom{m}{n}$ is the usual binomial coefficient. Normalize the ground-state wavefunction such that the smallest nonzero entry is unity. Then based on explorations on finite systems of size $L = 2n + 1$, Razumov and Stroganov have made the following conjectures (among others).

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Conjecture (Razumov and Stroganov [1]). *The largest element of the ground-state wavefunction is given by A_n, where*

$$A_n = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}. \tag{4}$$

Conjecture (Razumov and Stroganov [1]). *The 1- and 2-sums for the ground-state wavefunction are given by*

$$S^{(1)}_{2n+1} = (\sqrt{3})^n \mathcal{N}_n$$
$$S^{(2)}_{2n+1} = \mathcal{N}_n^2$$

where

$$\mathcal{N}_n = \left(\frac{\sqrt{3}}{2}\right)^n \frac{2\cdot 5\cdots(3n-1)}{1\cdot 3\cdots(2n-1)} A_n. \tag{5}$$

The numbers A_n form the well known sequence $1, 2, 7, 42, 429, 7436, 218\,348, \ldots$ of $n \times n$ alternating-sign matrices [4], or equivalently, the total number of descending plane partitions with largest part less than or equal to n (see, e.g., [5] and references therein). The 2-sum $S^{(2)}_L$ gives the squared norm of the ground-state wavefunction.

Two variations of the *XXZ* chain with such a special ground-state energy at $\Delta = -\frac{1}{2}$ are also known. Consider first the Hamiltonian (1), with twisted boundary conditions

$$\sigma^z_{L+1} = \sigma^z_1 \qquad \sigma^\pm_{L+1} = \mathrm{e}^{\pm\mathrm{i}\phi}\sigma^\pm_1 \tag{6}$$

where $\sigma^\pm = \sigma^x \pm \mathrm{i}\sigma^y$ are the standard raising and lowering operators. For $\Delta = -\cos\lambda$ and twist angle $\phi = 2\lambda$, with $\lambda = \pi/3$ the ground-state energy is again given by (2), this time for L even [2]. In this case the ground-state wavefunction is complex. Nevertheless, we also find some surprising connections with the numbers A_n. We choose the normalization such that the smallest elements have amplitude 1 and the wavefunction is invariant under the combined transformation of interchanging left and right and complex conjugation. Again, based only on exploration of ψ_0 for finite sizes, there are patterns to be observed. Our conjectures are the following.

Conjecture 1. *The 1-sum of the ground-state wavefunction of the twisted XXZ chain at $\Delta = -\frac{1}{2}$ is given by*

$$S^{(1)}_{2n} = 3^{n/2} A_n.$$

Conjecture 2. *The 2-sum of the ground-state wavefunction of the twisted XXZ chain at $\Delta = -\frac{1}{2}$ is given by*

$$S^{(2)}_{2n} = A_n^2.$$

Now consider the *XXZ* chain with the open boundary conditions [2, 6]

$$H = -\tfrac{1}{2}\left[\sum_{j=1}^{L-1}\left(\sigma^x_j\sigma^x_{j+1} + \sigma^y_j\sigma^y_{j+1} + \Delta\sigma^z_j\sigma^z_{j+1}\right) - \mathrm{i}\sin\lambda(\sigma^z_1 - \sigma^z_L)\right]. \tag{7}$$

This Hamiltonian maps directly to that of the Q-state Potts chain, with $\sqrt{Q} = 2\cos\lambda$. It is also quantum $U_q[sl(2)]$-invariant [7]. In the previous case the twisted boundary conditions (6) ensure that the ground state of the *XXZ* chain maps to that of the periodic Potts model. At $\Delta = -\frac{1}{2}$ the ground-state energy of (7) is given by $E_0 = -3(L-1)/4$ for both L even and odd [6] (see also [8, 9]). The ground-state wavefunction is again complex. We again choose the normalization such that the smallest elements have amplitude 1 and the wavefunction is invariant under the combined transformation of interchanging left and right and complex conjugation. For this case we are led to the following conjectures[3].

Conjecture 3. *The 1-sum of the ground-state wavefunction of the quantum invariant XXZ chain at* $\Delta = -\frac{1}{2}$ *is given by*

$$S_{2n}^{(1)} = 3^{n/2} A_{\mathrm{v}}(2n+1)$$
$$S_{2n-1}^{(1)} = 3^{(n-1)/2} N_8(2n)$$

where $A_{\mathrm{v}}(2n+1)$ *is the number of* $(2n+1) \times (2n+1)$ *vertically symmetric alternating-sign matrices ([10], theorem 2) given by*

$$A_{\mathrm{v}}(2n+1) = (-3)^{n^2} \prod_{i=1}^{2n+1} \prod_{j=1}^{n} \frac{3(2j-i)+1}{2j-i+2n+1} \tag{8}$$

and $N_8(2n)$ *is the number of cyclically symmetric transpose complement plane partitions [5, 11] given by*

$$N_8(2n) = \prod_{i=1}^{n-1} (3\mathrm{i}+1) \frac{(2\mathrm{i})!(6\mathrm{i})!}{(4\mathrm{i})!(4\mathrm{i}+1)!}. \tag{9}$$

Conjecture 4. *The 2-sum of the ground-state wavefunction of the quantum invariant XXZ chain at* $\Delta = -\frac{1}{2}$ *is given by*

$$S_{2n}^{(2)} = A_{\mathrm{v}}(2n+1)^2$$
$$S_{2n-1}^{(2)} = N_8(2n)^2.$$

Several comments are in order. In the past, wavefunctions in Bethe ansatz systems such as the *XXZ* chain have been traditionally ignored because of their unwieldly nature. Combined with the observations of Razumov and Stroganov for the periodic odd chain, we now see that there is some remarkable structure in the first two p-sums for the ground-state wavefunction of the antiferromagnetic *XXZ* chain at $\Delta = -\frac{1}{2}$. Presumably the conjectured results can be proved in each case via the Bethe ansatz form of the wavefunctions, which *are* known. Also known in each case is an explicit form of Baxter's Q-function [3, 8, 9], the zeros of which give the Bethe roots. We hope that our findings will spark further interest in their properties.

Even more remarkable is that the observed structure is related to alternating-sign matrices, or rather plane partitions. For the two cases considered here the trivial ground-state energy can be viewed as a result of the trivial representation in the related Temperley–Lieb–Jones algebra [2]. That may also have a bearing on the wavefunction. As noted above the wavefunctions are complex. However, they are real in the corresponding dense $\mathrm{O}(n)$ loop model [12, 13], in which closed loops carry fugacity $n = 2\cos\lambda$ (in this case $n = 1$). The numbers appearing in the first few ground-state wavefunctions for open and periodic boundary conditions are given in tables 1–3. In each case we normalize the wavefunctions such that the smallest element is unity[4]. Four further conjectures suggest themselves.

Conjecture 5. *For open boundary conditions, the largest element of the* $\mathrm{O}(1)$ *loop model wavefunction is given by* $N_8(2n)$ *for* $L = 2n$ *and* $A_{\mathrm{v}}(2n-1)$ *for* $L = 2n-1$.

[3] In this case we found Sloane's *On-Line Encyclopaedia of Integer Sequences* to be most helpful, see `http://www.research.att.com/~njas/sequences/`.

[4] Note that for the loop models, the upper limits in the p-sums defined in (3) may be different to their binomial counterparts in the *XXZ* chain.

Table 1. Ground-state wavefunctions of the O(1) loop model with open boundaries (L even). Note that by $\psi_0 = (11, 5, 4, 1)$ with multiplicity $(1, 2, 1, 1)$ we mean $\psi_0 = (11, 5, 5, 4, 1)$ etc.

L	n	ψ_0	Multiplicity	$S_{2n}^{(1)}$
2	1	(1)	(1)	1
4	2	(2, 1)	(1, 1)	3
6	3	(11, 5, 4, 1)	(1, 2, 1, 1)	26
8	4	(170, 75, 71, 56, 50, 30, 14, 6, 1)	(1, 2, 1, 2, 1, 1, 4, 1, 1)	646

Table 2. Ground-state wavefunctions of the O(1) loop model with open boundaries (L odd).

L	n	ψ_0	Multiplicity	$S_{2n-1}^{(1)}$
1	1	(1)	(1)	1
3	2	(1)	(2)	2
5	3	(3, 1)	(3, 2)	11
7	4	(26, 10, 9, 8, 5, 1)	(4, 2, 2, 2, 2, 2)	170

Table 3. Ground-state wavefunctions of the O(1) loop model with periodic boundaries.

L	n	ψ_0	Multiplicity	$S_{2n}^{(1)}$
2	1	(1)	(1)	1
4	2	(1)	(2)	2
6	3	(2, 1)	(2, 3)	7
8	4	(7, 3, 1)	(2, 8, 4)	42
10	5	(42, 17, 14, 6, 4, 1)	(2, 10, 5, 10, 10, 5)	429

Conjecture 6. *The 1-sum of the ground-state wavefunction of the* O(1) *loop model with open boundary conditions is given by*

$$S_{2n}^{(1)} = A_{\rm v}(2n+1)$$
$$S_{2n-1}^{(1)} = N_8(2n).$$

Conjecture 7. *For periodic boundary conditions and* $L = 2n$*, the largest element of the* O(1) *loop model wavefunction is given by* A_{n-1}*.*

Conjecture 8. *The 1-sum of the ground-state wavefunction of the* O(1) *loop model with periodic boundary conditions and* $L = 2n$ *is given by* $S_{2n}^{(1)} = A_n$*.*

In view of the relative simplicity of the loop model results, we also considered the loop version of the *XXZ* chain at $\Delta = -\frac{1}{2}$ for odd sites. The numbers appearing in the first few ground-state wavefunctions are given in table 4. Again we have normalized the wavefunctions such that the smallest element is unity. Two further conjectures are the following.

Conjecture 9. *For periodic boundary conditions and* $L = 2n - 1$*, the largest element of the* O(1) *loop model wavefunction is given by* A_{n-1}^2*.*

Conjecture 10. *The 1-sum of the ground-state wavefunction of the* O(1) *loop model with periodic boundary conditions and* $L = 2n - 1$ *is given by* $S_{2n-1}^{(1)} = \mathcal{N}_n^2$*.*

These are to be compared with those of Razumov and Stroganov [1] given above. An interesting comparison can be made between conjectures 7 and 9 for periodic boundary conditions with even and odd system sizes. On the one hand, for L even, the largest element

Table 4. Ground-state wavefunctions of the O(1) loop model with periodic boundaries (L odd). The multiplicities are divided by L.

L	n	ψ_0	Multiplicity	$S_{2n-1}^{(1)}$
1	1	(1)	(1)	1
3	2	(1)	(1)	3
5	3	(4, 1)	(1, 1)	25
7	4	(49, 14, 6, 1)	(1, 2, 1, 1)	588
9	5	(1764, 567, 525, 266, 150, 132, 49, 27, 8, 1)	(1, 1, 2, 2, 1, 1, 2, 2, 1, 1)	39 204

in the wavefunction is given by A_{n-1}, which we recall is also equivalent to the total number of descending plane partitions with largest part less than or equal to $n-1$. This quantity is also the number of totally symmetric self-complementary plane partitions in $\mathcal{B}(2n-2, 2n-2, 2n-2)$, a cubic box of linear size $2n-2$. On the other hand, for L odd, the largest element is given by A_{n-1}^2. However, A_{n-1}^2 is the number of cyclically symmetric self-complementary plane partitions in $\mathcal{B}(2n-2, 2n-2, 2n-2)$ [5]. Further, conjecture 5 states that for L even and open boundaries, the largest element in the wavefunction is given by the number of cyclically symmetric transpose complement plane partitions in $\mathcal{B}(2n, 2n, 2n)$. Such results may be a hint that the underlying relations are more directly with plane partitions rather than alternating-sign matrices.

The appearance of alternating-sign matrices may also be a remnant of the eigenspectrum symmetry between the points $\Delta = -\frac{1}{2}$ and $\frac{1}{2}$. After all, the point $\Delta = \frac{1}{2}$ corresponds to the ice model, which has a well documented connection with alternating-sign matrices with domain wall boundary conditions in the vertex formulation [5]. In particular, Kuperberg [10] has shown how to derive a number of results for various symmetry classes of alternating-sign matrices from the six-vertex model with different boundary conditions. We find that to be a very interesting paper.

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